B.Sc-part-I(H), paper-I Abstract Algebra 17-07-2020

Group.

Q.1 Define Group with ~~an~~ ~~examples~~ examples?

Soln: Group:→ Let o be a binary operation defined over a non-empty set G. Then the set G together with the operation 'o' satisfying the following postulates is called a 'group'.

If $a, b, c \in G$, then

I $a o b \in G$. (Closure axiom)

II The operation is associative,

that is $a o (b o c) = (a o b) o c$ (Associative axiom)

III There exists an element $e \in G$ such that

$$a o e = e o a = a$$

The element e is called an identity element in G. (identity axiom)

IV For every $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1} o a = a o a^{-1} = e$ (inverse axiom)

a^{-1} is called the inverse of a in G.

For examples. I The set of integers including zero forms a group with respect to addition.

II The set of real numbers with addition as operation forms a group.

Q.2. Define sub-group with ~~an~~ example.

Soln: Subgroup:→ Let (G, o) be any group and H any sub-set of G. If the set H together with the operation o forms a group (H, o), then (H, o) is a sub-group of (G, o).

For example. The additive group of even integers H is a sub-group of the additive group of integers G.

~~Theorem~~ Q. 3 Let (G, o) be a group, then 17-07-2020

(I) The identity element e is unique

(II) Every element $a \in G$ has unique inverse in G.

Proof: (I) Let e_1 and e_2 be two distinct identities of the group G. Then, by definition of identity, we have

$$e_1 o e_2 = e_1 \quad (\text{since } e_2 \text{ is identity})$$

and

$$e_1 o e_2 = e_2 \quad (\because e_1 \text{ is identity})$$

Hence, it follows that

$$e_1 = e_2$$

$\Rightarrow$ the identity is unique

(II) Let any element $a \in G$ have two inverses b and c, then we have

$$a o b = e = b o a$$

and

$$a o c = e = c o a$$

$$\therefore\ b = b o e = b o (a o c) = (b o a) o c \quad [\text{By associativity}]$$

$$= e o c = c$$

$$\Rightarrow b = c$$

Hence, $a \in G$ has unique inverse.

Q.4. If (G, o) is a group, then

(I) $(a^{-1})^{-1} = a \ \forall\ a \in G$ (II) $(a o b)^{-1} = b^{-1} o a^{-1}, \ \forall\ a, b \in G$.

Proof: I. Let $a \in G$, there exists an element $b \in G$ such that

$$a o b = b o a = e$$

From the symmetry of this result, we have

$$a^{-1} = b \quad \text{—— I}$$

and

$$b^{-1} = a \quad \text{—— II}$$

Putting the value of b in Eqn (II) we get

$$(a^{-1})^{-1} = a, \ \forall\ a \in G$$

II For all $a, b \in G$, we have

$$(a o b) o (b^{-1} o a^{-1}) = a o (b o b^{-1}) o a^{-1} \quad [\text{By associativity}]$$

$$= a o (e) o a^{-1} = (a o e) o a^{-1} \quad [\text{By associativity}]$$

$$= a o a^{-1} = e$$

Similarly, we can easily show that

$$(b^{-1} o a^{-1}) o (a o b) = e$$

Thus, $(a o b) o (b^{-1} o a^{-1}) = e = (b^{-1} o a^{-1}) o (a o b)$

Hence, it follows that $(a o b)^{-1} = b^{-1} o a^{-1}$

Proved